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Propagator, Sewing Rules, and Vacuum Amplitude for the Polyakov Point Particle with Ghosts

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We apply techniques developed for strings to the case of the spinless point particle. The Polyakov path integral with ghosts is used to obtain the propagator and one-loop vacuum amplitude. The propagator is shown to correspond to the Green's function for the BRST field theory in Siegel gauge. The reparametrization invariance of the Polyakov path integral is shown to lead automatically to the correct "trace log" result for the one-loop diagram, despite the fact that "naive sewing" of the ends of a propagator would give an incorrect answer. This type of failure of "naive sewing" is identical to that found in the string case. The present treatment provides, in the simplified context of the point particle, a pedagogical introduction to Polyakov path integral methods with and without ghosts.

1. INTRODUCTION

The Polyakov version (Polyakov, 1981; Green *et al.*, 1987) of the path integral for reparametrization-invariant systems is at present the most useful method for carrying out string-theoretic computations in a covariant manner. Although it is a first-quantized technique, it can be used to learn about the structure of covariantly second-quantized string theory, i.e., covariant string field theory, since the n-string correlation functions of the string field theory correspond to first-quantized Polyakov path integrals over worldsheets with n boundaries fixed in spacetime (Witten, 1986; Kaku, 1988). Now, the most natural formulation of covariant string field theory (Siegel,

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1986) makes use of second-quantized fields which are functionals, not only of the spacetime coordinates $X^{\mu}(\sigma)$ of the string-at-a-single-time, but also of (first-quantized) ghost and antighost coordinates $c(\sigma)$, $b(\sigma)$. Therefore, the corresponding first-quantized path integral must also be a functional of ghost and antighost coordinates at the fixed boundaries of the worldsheet.

In a recent series of papers (Ordóñez *et al.*, 1987*a–c*) we have developed techniques for introducing the required ghost dependence into the Polyakov path integral for bosonic strings, and have used these techniques to evaluate the propagators for open and closed strings in conformal gauge (orthogonal coordinates on the string worldsheet). The amplitudes thus obtained are found to correspond to the Green's functions (tree-level two-string functions) of the covariant field theory in the Siegel gauge (Siegel, 1986). The propagators themselves cannot be "sewn" together, but the "Schwinger–DeWitt proper-time" amplitudes (DeWitt, 1964; Brown, 1977), of which the actual propagators are the integrals, do sew together properly. However, this "sewing rule" fails when we attempt to tie together the ends of a single propagator to make a closed loop, since it gives, incorrectly, the trace of the propagator, rather than the correct result, the trace of the *logarithm* of the inverse propagator (Ordóñez *et al.*, 1987*c*). On the other hand, applying the Polyakov prescription directly to the loop (Polchinski, 1986) does give the correct answer.

The properties of the Polyakov path integral with ghosts which we have singled out for mention in the above paragraph seem not to be peculiar to strings, but rather, to be generic aspects of reparametrization-invariant systems quantized through the Polyakov prescription. In the present paper we apply the same methods to the quantization of the spinless relativistic point particle, and verify that these properties are indeed present there as well. [On the other hand, the fact that the correct modular integration region is not obtained when the logarithm of the closed-string propagator is sewn to make a torus (Ordóñez *et al.*, 1987*c*) is clearly a "stringy" feature with no point-particle analogue.]

The point-particle propagator without ghosts has been computed à la Polyakov by Cohen *et al.* (1986), Govaerts (1988), and Henty *et al.* (1988). (See also Mannheim, 1986.) Here, as in Ordóñez *et al.* (1987*a–c*), we follow the treatment of gauge fixing of Chaudhuri *et al.* (1987). We have attempted to present in detail aspects of the computations involving subtleties of a conceptual nature, while at the same time avoiding details pertaining only to subtleties of extreme mathematical rigor. Moreover, the complications due to Weyl invariance of the string action are, of course, absent in the particle case. It is thus our hope that the present paper can also serve as a pedagogical introduction to techniques used in gauge fixing the Polyakov path integral for strings (Weinberg, 1987).

2. THE PROPAGATOR

The action for a spinless relativistic point particle of mass m is usually written as

$$S = m \int_0^1 d\tau \, [\dot{X}^\mu(\tau)\dot{X}^\mu(\tau)]^{1/2} \tag{2.1}$$

The index μ in (1) is summed from 1 to D, the number of spacetime dimensions. $X^\mu(\tau)$ is thus a D-dimensional vector parametrized by the single real parameter τ; τ will always be taken to run from 0 to 1. The dot over $X^\mu(\tau)$ denotes differentiation with respect to τ.

We will work in a Euclideanized spacetime with the flat spacetime metric $\eta_{\mu\nu} = \text{diag}(1, 1, \ldots, 1)$, so we will not bother about raising or lowering spacetime indices μ, ν, but will write them all "up" (on the other hand, we will pay careful attention to the indices, to be introduced shortly, which denote tensor type on the particle worldline rather than in spacetime, because the transformation properties of dynamical quantities under change of worldline coordinates will be crucial to the analysis).

The particle worldline $X^\mu(\tau)$ is a one-dimensional manifold with boundary, and τ serves as a coordinate on this manifold. With respect to general coordinate transformations on the worldline (for which we will also use, interchangeably, the terms "reparametrizations" and "diffeomorphisms"), i.e.,

$$\tau \to \tilde{\tau} = \tilde{\tau}(\tau) \tag{2.2}$$

$X^\mu(\tau)$ is a scalar and $\dot{X}^\mu(\tau)$ is a covariant rank-one tensor. Of course, the action S is also a scalar under reparametrizations.

We introduce a covariant rank-two tensor on the worldline, the "metric tensor" $g_{ab}(\tau)$. The "worldline indices" a, b take on only a single value and thus really serve only to denote tensor character. Consider now the action

$$\begin{aligned} S_P &= \tfrac{1}{2} \int_0^1 d\tau \, g^{1/2}(g^{ab}\partial_a X^\mu \partial_b X^\mu + m^2) \\ &= \tfrac{1}{2} \int_0^1 d\tau \, g^{1/2}\left(g^{ab} \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\mu}{\partial \tau} + m^2\right) \\ &= \tfrac{1}{2} \int_0^1 d\tau \, g^{1/2}(g^{ab}\dot{X}^\mu \dot{X}^\mu + m^2) \end{aligned} \tag{2.3}$$

where

$$g \equiv \det g_{ab} = g_{ab} \tag{2.4}$$

$$g^{ab} = (g_{ab})^{-1} = g^{-1} \tag{2.5}$$

If g_{ab} is treated as a dynamical field along with X^{μ}, then S_P is equivalent to S at the classical level; that is, it gives the same equation of motion for X^{μ}, provided g_{ab} satisfies its equation of motion. The equation of motion for g_{ab} is

$$\frac{\delta S}{\delta g_{ab}(\tau)}=0 \tag{2.6}$$

or

$$g_{ab}(\tau)=\frac{1}{m^2}\dot{X}^{\mu}(\tau)\dot{X}^{\mu}(\tau) \tag{2.7}$$

Using (2.1) and (2.3)-(2.5), we see that, when (2.7) holds, $S_P=S$.

To obtain the quantum mechanical amplitude for a particle at spacetime point X_i^{μ} to propagate to spacetime point X_f^{ν}, we perform the path integral

$$\langle X_f | X_i \rangle = \int_{X_i^{\mu}}^{X_f^{\nu}} \frac{Dg_{ab}DX}{V_{GC}} \exp(-S_P) \tag{2.8}$$

The path integral in (2.8) is over all spacetime-vector-valued functions $X^{\mu}(\tau)$ satisfying

$$X^{\mu}(0)=X_i^{\mu}, \qquad X^{\mu}(1)=X_f^{\mu} \tag{2.9}$$

and over all possible worldline-tensor-valued functions $g_{ab}(\tau)$ (just a single function of τ, of course). V_{GC} is a normalization factor, about which more will be said below. The path integrand is $\exp(-S_P)$ instead of $\exp(iS_P)$ because, as mentioned above, we are working in the Euclideanized theory.

As was the case with the action S, the action S_P is invariant under one-dimensional general coordinate transformations. Because of this, the path integration in (2.8) "overcounts" by an infinite amount, since for any given metric $g_{ab}(\tau)$ and spacetime trajectory $X^{\mu}(\tau)$ included in the path integral there will also be included an infinite number of metrics and spacetime trajectories related to the given metric and trajectory by general coordinate transformations.

This problem is essentially the same as the problem, in quantum electrodynamics or Yang–Mills theories, of overcounting by integrating over fields related to each other by gauge transformations. The solution we employ here is a version of the solution in the gauge-theory case (Faddeev and Popov, 1967; Abers and Lee, 1973).

The key fact is the following: one can find a set of so-called "fiducial metrics," metrics *not* related to each other by general coordinate transformations but having the property that *every* metric can be obtained from one of the fiducial metrics by a specific general coordinate transformation. (We will demonstrate shortly the construction of a particular fiducial set.) So, integrating over the space of all metrics is equivalent to: fixing a fiducial metric; integrating over all diffeomorphisms, so as to cover the subspace of metrics related by diffeomorphisms to the chosen fiducial metric; and then integrating over all metrics in the set of fiducial metrics.

Let us denote, in a schematic fashion, fiducial metrics by the symbol $\hat{g}$ and diffeomorphisms by the symbol η. (The notation will be made more concrete shortly.) Then we can rewrite the right-hand side of (2.8) as

$$\int \frac{Dg_{ab}\, DX}{V_{GC}} \exp(-S_P[X, g_{ab}]) = \int \frac{D\hat{g}\, DX\, D\eta}{V_{GC}} \exp(-S_P[X, \hat{g}, \eta]) \quad (2.10)$$

One might think that the path integration in (2.10) still overcounts, since we are still integrating over *all* configurations $X^\mu(\tau)$. But (2.10) is correct as it stands; see the discussion in the Appendix.

In writing (2.10) as we have, we have implied that the measures for path integration $\int D\hat{g}$ and $\int DX$ can be defined in a diffeomorphism-invariant manner; otherwise, it would make no sense to perform these integrations *after* $\int D\eta$. In what follows, we will therefore be obliged to define these measures in a general-coordinate-invariant manner. (On the other hand, $\int D\eta$ does in some cases depend on $\hat{g}$; e.g., in the loop computation in Section 5.)

The action S_P, being the integral over the worldline of a worldline density, is completely unchanged by diffeomorphisms:

$$S_P[X, \hat{g}, \eta] = S_P[X, \hat{g}] \quad (2.11)$$

Integration over all diffeomorphisms[3] is precisely what we mean by V_{GC}:

$$V_{GC} = \int D\eta \quad (2.12)$$

Using (2.10)-(2.12) in (2.8), we obtain

$$\langle X_i | X_f \rangle = \int D\hat{g}\, DX \exp(-S_P[X, \hat{g}]) \quad (2.13)$$

To implement this concretely, we must find a specific set of fiducial metrics. Although the space of metrics (i.e., the space of functions of $\tau \in [0, 1]$) is infinite-dimensional, the set of fiducial metrics is one-dimensional, i.e., parametrized by a single real parameter, which we denote by λ, and refer to as the "Teichmuller parameter." We are free to choose the functional dependence of the fiducial metrics on λ; the most convenient choice is

$$\hat{g}_{ab} = \lambda^2, \qquad 0 < \lambda < \infty \quad (2.14)$$

[3]In general, we should include in (2.12) a factor of the square root of the determinant of a metric on the space of diffeomorphisms; however, in the coordinates (2.23), (2.40) which we will use for the space of diffeomorphisms, this factor is unity. See Ordóñez *et al.*, 1988.

so

$$\hat{g}^{ab} = \lambda^{-2} \tag{2.15}$$

$$\hat{g} \equiv \det \hat{g}_{ab} = \lambda^2 \tag{2.16}$$

We note that the fiducial metric (2.14) is independent of position τ on the worldline. λ is just the geometrical length of the worldline

$$\lambda = \int_0^1 d\tau\, \hat{g}^{1/2} \tag{2.17}$$

manifestly unchanged under change of worldline coordinates.[4]

Since the space of metrics is infinite-dimensional while the space of fiducial metrics is one-dimensional, the space of general coordinate transformations (which generate the entire space of metrics starting from a fiducial set) must be an infinite-dimensional space. This is in fact the case; writing the transformations (2.2) as

$$\tau \to \bar{\tau}(\tau) = \tau + \eta^a(\tau) \tag{2.18}$$

we see that we can parametrize general coordinate transformations by the (infinite-dimensional) space of functions $\eta^a(\tau)$.

We view the coordinate transformations (2.18) as active transformations, mapping points on the worldline to different points on the worldline. For infinitesimal transformations, η^a is a worldline vector field. The actions of such transformations on the worldline scalars $X^\mu(\tau)$ and the worldline tensors $g_{ab}(\tau)$ are, respectively,

$$\begin{aligned} X^\mu(\tau) \to \bar{X}^\mu(\tau) &= X^\mu(\tau) + \eta^a(\tau)\partial_a X(\tau) \\ &= X^\mu(\tau) + \eta^a(\tau)\dot{X}(\tau) \end{aligned} \tag{2.19}$$

$$\begin{aligned} g_{ab}(\tau) \to \bar{g}_{ab}(\tau) &= g_{ab}(\tau) + \nabla_a \eta_b(\tau) + \nabla_b \eta_a(\tau) \\ &= g_{ab}(\tau) + 2\nabla_a \eta_b(\tau) \end{aligned} \tag{2.20}$$

(∇_a is the one-dimensional covariant derivative; we will not need its explicit form.) These active transformations are referred to as "Lie dragging" (Schutz, 1980; Wald, 1984; Weinberg, 1972).

From (2.19) we see that, if η^a does not vanish at the ends of the worldline, it generates physical motions of the ends of the worldline. However, the amplitude (2.8) is a path integral over worldlines with *fixed*

[4]For every $g_{ab}(\tau)$, there is a unique general coordinate transformation taking $g_{ab}(\tau)$ to $\hat{g}_{ab} = \lambda^2 = [\int_0^1 d\tau (g^{1/2})]^2$. Specifically, $\hat{g}_{ab} = (d\tau/d\tau')^2 g_{ab}(\tau) = (d\tau'/d\tau)^{-2} g_{ab}(\tau)$, where $\tau'(\tau) = [\int_0^1 d\bar{\tau}(g^{1/2})]^{-1} \int_0^\tau d\bar{\tau}\, g^{1/2}$. We see that the requirement of fixed endpoints is satisfied: $\tau'(0) = 0$, $\tau'(1) = 1$. So the set of metrics (2.14) is in fact a fiducial set.

endpoints. The reparametrizations over which we want to integrate are thus generated by vector fields satisfying

$$\eta^a(0) = \eta^a(1) = 0 \tag{2.21}$$

However, not all vectors satisfying (2.21) generate new metrics from the fiducial ones. From (2.20) we see that we want only vectors satisfying

$$\nabla_a \eta_b(\tau) \neq 0 \tag{2.22}$$

for at least some τ.

So, the space of metrics over which we want to path integrate has as coordinates the Teichmuller parameter λ and the vectors $\eta^a(\tau)$ satisfying (2.21) and (2.22). Instead of $\eta^a(\tau)$, we can equivalently use η_m, the expansion coefficients of $\eta^a(\tau)$ in terms of some suitable set of eigenfunctions $\psi^a_m(\tau)$:

$$\eta^a(\tau) = \sum_m \eta_m \psi^a_m(\tau) \tag{2.23}$$

where the eigenfunctions satisfy

$$\psi^a_m(0) = \psi^a_m(1) = 0 \tag{2.24}$$

$$\nabla_a \psi_{bm}(\tau) \neq 0 \tag{2.25}$$

so that $\eta^a(\tau)$ automatically satisfies (2.21) and (2.22). The integral "over all metrics" is thus

$$\int Dg = \int_0^\infty d\lambda \prod_m{}' \int_{-\infty}^\infty d\eta^a_m \, \mu(\lambda, \eta) \tag{2.26}$$

where the prime is a reminder to delete from the product values of m violating (2.25). $\mu(\lambda, \eta)$ is an integration measure in the space of metrics. If we were dealing with integration over the usual type of manifold, we would take μ to be the square root of the determinant of the metric tensor, evaluated in a basis corresponding to the coordinate system we are using. We will therefore construct a metric on the space of metrics $g_{ab}(\tau)$—a "supermetric"—and use the square root of its determinant for the measure.

A metric, in any space, is a symmetric bilinear map from tangent vectors to numbers. In addition to the coordinates (λ, η_m) we have already described, $g_{ab}(\tau)$ itself serves as a coordinate on the space of metrics. To obtain the components of a tangent vector in the space of metrics, we consider a parametrized curve in this space, and use the coordinates $g_{ab}(\tau)$. Denote the parameter by a subscript α or β and the curve by $g_{ab}(\tau)_\alpha$; the components of the tangent vector at the point $g_{ab}(\tau)_{\alpha_0}$ are

$$V_{ab}(\tau)[g_{\alpha_0}]_{(\alpha)} \equiv \frac{d}{d\alpha}\bigg|_{\alpha=\alpha_0} g_{ab}(\tau)_\alpha \tag{2.27}$$

That is, V points along the curve we have parametrized by α [this is indicated by the subscript (α)] at the point where the parameter has the value α_0, that is, the point $g_{ab}(\tau)_{\alpha_0}$.

The most general supermetric which is only a function of the $g_{ab}(\tau)$ (and not their derivatives) and which respects one-dimensional general coordinate invariance is

$$G^{ab,cd}(\tau_1, \tau_2) = g^{1/2}(\tau_1)[(g^{ab}(\tau_1)g^{cd}(\tau_1) + Cg^{ac}(\tau_1)g^{bd}(\tau_1)]\delta(\tau_1 - \tau_2)$$
$$= (1+C)[g(\tau_1)]^{-3/2}\,\delta(\tau_1 - \tau_2) \tag{2.28}$$

using (2.4) and (2.5). C is an arbitrary constant, which we will, for notational convenience, set to zero. (This is just an adjustment of the overall scale of the path integral.) The reparametrization-invariant inner product is

$$(V_{(\alpha)}, GV_{(\beta)}) = \int_0^1 d\tau_1 \int_0^1 d\tau_2\; V_{ab}(\tau_1)_{(\alpha)} G^{ab,cd}(\tau_1, \tau_2)\, V_{cd}(\tau_2)_{(\beta)}$$
$$= \int_0^1 d\tau\; V_{ab}(\tau)_{(\alpha)}[g(\tau)]^{-3/2}\, V_{cd}(\tau)_{(\beta)} \tag{2.29}$$

As discussed earlier, we prefer to use the coordinates (λ, η_m) rather than $g_{ab}(\tau)$, so as to be able to separate out the redundant integration over metrics related by diffeomorphisms; so we need the components of the supermetric in the (λ, η_m) coordinate system. Let us first obtain these components at the point $\lambda = \lambda_0$, $\eta_m = 0$ for all m. Since $\eta_m = 0$, $g_{ab}(\tau)$ is the fiducial metric,

$$g_{ab}(\tau)\big|_{\lambda=\lambda_0, \eta_m=0} = \lambda_0^2 \tag{2.30}$$

A vector pointing in the direction of increasing λ has components

$$V_{ab}(\tau)_{(\lambda)} = \frac{\partial g_{ab}}{\partial \lambda}\bigg|_{\lambda=\lambda_0, \eta_m=0} = 2\lambda_0 \tag{2.31}$$

Using (2.20), (2.23), and (2.30), we see that a vector pointing in the direction of increasing η_m is

$$V_{ab}(\tau)_{(m)} = \frac{\partial g_{ab}}{\partial \eta_m}\bigg|_{\lambda=\lambda_0, \eta_m=0} = 2\,\partial_a\psi_{bm}(\tau) = 2\lambda_0^2\,\partial_a\psi^b_m(\tau) \tag{2.32}$$

(At $\eta_m = 0$ we can replace covariant derivatives by ordinary derivatives, since the fiducial metric is a constant.) Choose the ψ^a_m to be eigenfunctions of the one-dimensional Laplacian,

$$\partial_a\partial^a\psi^a_m(\tau) = -\Lambda_m\psi^a_m(\tau) \tag{2.33}$$

and to satisfy the reparametrization-invariant normalization condition

$$\int_0^1 d\tau\, g^{1/2} \psi^a_m \psi_{an} = \delta_{mn} \tag{2.34}$$

Using (2.24) and (2.29)-(2.34), we find that, at $\lambda = \lambda_0$, $\eta_m = 0$,

$$G_{\lambda\lambda} = (V_{(\lambda)}, GV_{(\lambda)}) = 4\lambda_0^{-1} \tag{2.35}$$

$$G_{\lambda m} = G_{m\lambda} = (V_{(\lambda)}, GV_{(m)}) = 0 \tag{2.36}$$

$$G_{mn} = (V_{(m)}, GV_{(n)}) = \delta_{mn}\Lambda_n \tag{2.37}$$

To obtain vector fields and supermetrics at other points in the space of metrics, i.e., those points where $\eta_m \neq 0$ for some m, we perform diffeomorphisms on the V's and G's; however, the inner product (2.29) is unchanged under diffeomorphisms. So the expressions (2.35)-(2.37) for the components of G are valid at all points in the space of metrics. Since G is diagonal, we obtain from (2.35)-(2.37)

$$\mu(\lambda, \eta) = \mu(\lambda) \equiv (\det' G)^{1/2} = 2\lambda^{-1/2} \prod_m{}' \Lambda_m^{1/2} \tag{2.38}$$

Using (2.8), (2.10), (2.11), (2.26), and (2.38), and dropping irrelevant overall constants (a practice we will employ with no further warning throughout the rest of this paper), we obtain

$$\langle X_f | X_i \rangle = \int_0^\infty d\lambda \left[\frac{\int \prod'_m d\eta^a_m}{V_{GC}}\right] \lambda^{-1/2} \left(\prod_m{}' \Lambda_m^{1/2}\right) \int_{X_i}^{X_f} DX \exp(-S_P[X, \hat{g}]) \tag{2.39}$$

In general, we cannot at this point simply set the quantity in square brackets in (2.39) to one, since V_{GC} is the volume of *all* diffeomorphisms, whereas the numerator is an integral over all diffeormorphisms *except* zero modes, i.e., those modes violating (2.25). (Nor can we simply start over using a normalization factor $V'_{GC} \equiv \int \prod'_m d\eta^a_m$; this would violate reparametrization invariance. The different eigenfunctions ψ^a_m, including any zero modes, are mixed up among each other under diffeomorphisms; hence a factor of V'_{GC} is not a one-dimensional-diffeomorphism invariant.)

However, in the case we are currently considering, there are no zero-mode ψ^a_m, because of the boundary condition (2.24). With the normalization (2.34),

$$\psi^a_m = 2^{1/2} \lambda^{-3/2} \sin(\pi m \tau) \tag{2.40}$$

So, in this particular case, it happens that

$$\int \prod_m{}' d\eta^a_m = \int \prod_m d\eta^a_m = V_{GC} \tag{2.41}$$

and (2.39) becomes

$$\langle X_f | X_i \rangle = \int_0^\infty d\lambda\, \lambda^{-1/2} \left(\prod_m{}' \Lambda_m \right)^{1/2} \int_{X_i}^{X_f} DX \exp(-S_P[X, \hat{g}]) \quad (2.42)$$

We next perform the path integral over X. Using (2.14)-(2.16) in (2.3), we obtain

$$S_P = \tfrac{1}{2} \int_0^1 d\tau\, (\lambda^{-1} \dot{X}^\mu \dot{X}^\mu + \lambda m^2) \quad (2.43)$$

Following the usual path-integration method (Feynman and Hibbs, 1965), write $X^\mu(\tau)$ as the sum of a classical part $\bar{X}^\mu(\tau)$ and a quantum part $\tilde{X}^\mu(\tau)$:

$$X^\mu(\tau) = \bar{X}^\mu(\tau) + \tilde{X}^\mu(\tau) \quad (2.44)$$

$\bar{X}^\mu(\tau)$ is the solution to the classical equations of motion,

$$\frac{d^2}{d\tau^2} \bar{X}(\tau) = 0 \quad (2.45)$$

interpolating between the initial and final configurations

$$X^\mu(0) = X_i^\mu, \qquad X^\mu(1) = X_f^\mu \quad (2.46)$$

Since the value of an integral from $-\infty$ to $+\infty$ will be unchanged if we redefine the integration variable by adding a constant to it, we can shift $X^\mu(\tau)$ in the path integral by $\bar{X}^\mu(\tau)$ at each point τ:

$$\int DX \to \int D\tilde{X} \quad (2.47)$$

From (2.9), (2.44), and (2.46), we see that $\tilde{X}^\mu(\tau)$ satisfies

$$\tilde{X}^\mu(0) = \tilde{X}^\mu(1) = 0 \quad (2.48)$$

If we expand $\tilde{X}^\mu(\tau)$ in terms of eigenfunctions of the worldline Laplacian,

$$\tilde{X}^\mu(\tau) = \sum_n \tilde{X}_n^\mu f_n(\tau) \quad (2.49)$$

where

$$\partial_a \partial^a f_n(\tau) = -\Lambda_n f_n(\tau) \quad (2.50)$$

$$\int_0^1 d\tau\, \lambda f_m(\tau) f_n(\tau) = \delta_{mn} \quad (2.51)$$

the eigenvalues Λ_n in (2.51) will be the same as the vector eigenvalues in (2.33). Using (2.43)-(2.51), we obtain

$$\int_{X_i}^{X_f} DX \exp(-S_P) = \exp(-\bar{S}) \prod_{\mu=1}^{D} \prod_m{}' \int_{-\infty}^{\infty} d\tilde{X}_m^\mu \exp\left(-\tfrac{1}{2} \sum_n{}' \Lambda_n \tilde{X}_n^\mu \tilde{X}_n^\mu \right) \quad (2.52)$$

[We could have put in a measure factor with the $\tilde{X}_m^\mu$ integration in (2.52), as we did in the g_{ab} integral:

$$\int D\tilde{X} \to \prod_{\mu=1}^{D} \prod_m \int_{-\infty}^{\infty} d\tilde{X}_m^\mu\, \mu_x \quad (2.53)$$

with μ_X the square root of the determinant, in the $f_m(\tau)$ basis, of the X-supermetric γ,

$$\gamma(\tau, \tau')_{\mu\nu} = \text{const} \cdot g^{1/2}\delta(\tau - \tau')\eta_{\mu\nu} \tag{2.54}$$

Using (2.51), however, we can see that $\gamma_{m,n}$ will be independent of λ; thus $\mu_X = (\det \gamma_{m,n})^{1/2}$ would contribute an uninteresting overall multiplicative constant to the amplitude.]

The quantity in the first exponential in (2.52) is the action (2.43) evaluated at the classical path $\bar{X}^\mu(\tau)$:

$$\bar{S} = \frac{1}{2\lambda}(X_i^\mu - X_f^\mu)(X_i^\mu - X_f^\mu) + \frac{\lambda m^2}{2} \tag{2.55}$$

The primes on the product and on the sum in the second exponential in (2.52) are reminders that the terms with $\Lambda_n = 0$ should be deleted. Again, this is an irrelevant reminder in our present computation; as mentioned previously, the eigenspectrum of $\tilde{X}^\mu(\tau)$ is the same as that of $\eta^a(\tau)$, and thus has no zero mode. The eigenfunctions are [using (2.48) and (2.51)]

$$f_n(\tau) = 2^{1/2}\lambda^{-1/2}\sin(\pi n\tau), \qquad n = 1, 2, \ldots \tag{2.56}$$

while, using (2.15),

$$\partial_a \partial^a = \lambda^{-2}\, \partial^2/\partial\tau^2 \tag{2.57}$$

so

$$\Lambda_n = \pi^2 n^2/\lambda^2, \qquad n = 1, 2, \ldots \tag{2.58}$$

Using (2.52), (2.55), and (2.58) in (2.42) and performing the Gaussian integrals (Feynman and Hibbs, 1965) over the $\tilde{X}_m^\mu$ (recall that μ runs from 1 to D), we obtain

$$\langle X_f | X_i \rangle = \int_0^\infty d\lambda\, \lambda^{-1/2} \left(\prod_{m=1}^\infty \frac{\pi^2 m^2}{\lambda^2} \right)^{1/2} \left(\prod_{n=1}^\infty \frac{\pi^2 n^2}{\lambda^2} \right)^{-D/2} \exp(-\bar{S}) \tag{2.59}$$

We evaluate the infinite products in (2.59) by zeta-function regularization (Brink and Nielsen, 1973; Hawking, 1977; Candelas and Raine, 1977):

$$\begin{aligned}
\prod_{m=1}^\infty (am^b) &= \prod_{m=1}^\infty \exp\log(am^b) \\
&= \exp \sum_{m=1}^\infty (\log a + b \log m) \\
&= \exp\left(\log a \sum_{m=1}^\infty 1 + b \sum_{m=1}^\infty \log m \right) \\
&= \exp\left(\log a \cdot \lim_{s\to 0} \sum_{m=1}^\infty m^{-s} - b \cdot \lim_{s \to 0} \frac{d}{ds} \sum_{m=1}^\infty m^{-s} \right)
\end{aligned} \tag{2.60}$$

The sums in (2.60) converge, provided the real part of s is sufficiently large; the first, to the Riemann zeta function,

$$\zeta(s)=\sum_{m=1}^{\infty} m^{-s} \tag{2.61}$$

and the second to its derivative $\zeta'(s)=(d/ds)\zeta(s)$. These functions, in turn, are defined at $s=0$; in particular,

$$\zeta(0)=-\tfrac{1}{2}, \qquad \zeta'(0)=-\tfrac{1}{2}\log(2\pi) \tag{2.62}$$

Using (2.61) and (2.62) in (2.60), we obtain

$$\prod_{m=1}^{\infty} (am^b)=a^{-1/2}(2\pi)^{b/2} \tag{2.63}$$

So, finally, (2.59) becomes

$$\langle X_f | X_i \rangle = \int_0^{\infty} d\lambda\, \lambda^{-D/2} \exp\left[-\frac{1}{2\lambda}(X_i^{\mu}-X_f^{\mu})(X_i^{\mu}-X_f^{\mu})-\frac{\lambda m^2}{2}\right] \tag{2.64}$$

3. GHOSTS AND BRST FIELD THEORY

As mentioned earlier, the fields in covariant string field theory are functionals not only of the spacetime coordinates $X^{\mu}(\tau)$, but also of Grassmann-valued (anticommuting) ghost coordinates. To establish the connection between the first-quantized formalism (Polyakov path integral) and the field theory, ghost dependence must also be introduced into the former. This has been done in Ordóñez *et al.* (1987*a*–*c*); here we illustrate the technique in the (calculationally much simpler) case of the point particle. We first return to equation (2.59), and recall that the term

$$\Delta_{FP} \equiv \left[\prod_{m=1}^{\infty} \frac{\pi^2 m^2}{\lambda^2}\right]^{1/2} \tag{3.1}$$

is the volume factor corresponding to integration over diffeomorphisms:

$$\Delta_{FP}=(\det G_{mn})^{1/2} \tag{3.2}$$

It thus corresponds to the Faddeev–Popov factor in gauge theories and, as in gauge theories, it can be represented as a Grassmann path integral.

Define the ghost action

$$S_{\text{gh}} \equiv -\int_0^1 d\tau\, [g(\tau)]^{1/2} c^a(\tau) \nabla^b b_{ab}(\tau) \tag{3.3}$$

$c^a(\tau)$ is a worldline vector, $b_{ab}(\tau)$ a rank-two worldline tensor. Both are anticommuting c-numbers (Grassmann variables). S_{gh} is unchanged under the action of diffeomorphisms $\eta^a(\tau)$, and the same argument previously applied to other factors in the path integrand applies here as well. That is, if we denote by c_η, b_η the ghosts subjected to diffeomorphisms generated by $\eta^a(\tau)$, then

$$S_{gh}[c_\eta, b_\eta, g] = S_{gh}[c_\eta, b_\eta, \hat{g}(\lambda), \eta] = S_{gh}[c, b, \hat{g}] \tag{3.4}$$

Using (2.14)-(2.16), (3.3), and the final member of (3.4), we have

$$S_{gh} = -\lambda^{-1} \int_0^1 d\tau\, c^a \frac{\partial}{\partial\tau} b_{ab} \tag{3.5}$$

Impose the boundary conditions

$$c^a(0) = c^a(1) = 0 \tag{3.6}$$

$$\frac{\partial}{\partial\tau} b_{ab}(\tau)\Big|_0 = \frac{\partial}{\partial\tau} b_{ab}(\tau)\Big|_1 = 0 \tag{3.7}$$

and expand $c^a(\tau)$ and $b_{ab}(\tau)$ as

$$c^a(\tau) = \sum_{n=1}^{\infty} \left[\frac{2}{\lambda^3}\right]^{1/2} \sin(\pi n\tau) c_n \tag{3.8}$$

$$b_{ab}(\tau) = \sum_{n=0}^{\infty} [\lambda^3 2^{1-\delta_{m,0}}]^{1/2} \cos(\pi n\tau) b_n \tag{3.9}$$

Using (3.5), (3.8), and (3.9), we obtain

$$S_{gh} = \sum_{n=1}^{\infty} \frac{n\pi}{\lambda} c_n b_n \tag{3.10}$$

The usual rules of Grassmann integration (DeWitt, 1984; Ramond, 1981) give

$$\Delta_{FP} = \int Dc\, \tilde{D}b\, e^{-S_{gh}} \tag{3.11}$$

where

$$\int Dc \equiv \int \prod_{m=1}^{\infty} dc_m \tag{3.12}$$

$$\int \tilde{D}b \equiv \int \prod_{m=1}^{\infty} db_m \tag{3.13}$$

[The tilde in (3.13) is a reminder that the $m = 0$ mode is not included even though allowed by the boundary condition (3.7). S_{gh} is independent of b_0, so, including an integral over b_0 would cause the path integral on the right-hand side of (3.11) to vanish.]

We could use the representation (3.11) of Δ_{FP} in the expression (2.59) for $\langle X_f | X_i \rangle$, but this would just give us another way of describing the same amplitude. We want an amplitude which depends on the values of ghost fields at $\tau = 0, 1$, in addition to the values of $X^\mu(\tau)$ at the endpoints. So we must modify equation (3.11) so that, instead of integrating over all possible b's and c's, we keep the end values fixed. Since we have already imposed (3.6), we will require

$$b_{ab}(0) = b_i, \qquad b_{ab}(1) = b_f \tag{3.14}$$

and define a new path integral A_{gh} by inserting δ functions to enforce (3.14) in the path integral (3.11):

$$A_{\text{gh}}(b_f, b_i) \equiv \int Dc\, \tilde{D}b\, \delta(b_{ab}(1) - b_f)\, \delta(b_{ab}(0) - b_i)\, e^{-S_{\text{gh}}} \tag{3.15}$$

With regard to Grassmann integration, the Dirac delta function of a variable is the variable itself. Therefore,

$$A_{\text{gh}}(b_f, b_i) = \int Dc\, \tilde{D}b\, [b_{ab}(1) - b_f][b_{ab}(0) - b_i]\, e^{-S_{\text{gh}}} \tag{3.16}$$

However, a nonvanishing term in (3.16) must include an equal number of c_m and b_n factors; so, keeping in mind (3.8) and (3.9),

$$A_{\text{gh}}(b_f, b_i) = b_f b_i \int Dc\, \tilde{D}b\, e^{-S_{\text{gh}}} = b_f b_i \Delta_{FP} \tag{3.17}$$

Finally, we define the BRST propagator (transition amplitude with ghosts) by substituting $A_{\text{gh}}(b_f, b_i)$ for Δ_{FP} in (2.59):

$$\begin{aligned} \langle X_f, b_f | X_i, b_i \rangle &\equiv \int_0^\infty d\lambda\, \lambda^{-1/2} A_{\text{gh}}(b_f, b_i) \left(\prod_{m=1}^{\infty} \frac{\pi^2 n^2}{\lambda^2} \right)^{-D/2} \exp(-\bar{S}) \\ &= b_f b_i \langle X_f | X_i \rangle \end{aligned} \tag{3.18}$$

Clearly, if we integrate over b_f and b_i we recover the original amplitude:

$$\int db_i\, db_f\, \langle x_f, b_f | X_i, b_i \rangle = \langle X_f | X_i \rangle \tag{3.19}$$

So, for the particle, the propagator with ghosts is related to the propagator without ghosts in a rather trivial manner. In the case of the string, the analogous relation is less trivial and the corresponding computation far more involved (Ordóñez *et al.*, 1987*a*).

Siegel (1986) has described a BRST field theory for the point particle. In our notation, the second-quantized field in this theory is $\Phi(X, b)$ or, Taylor-expanding in b,

$$\Phi(X, b) = \psi(X) + b\phi(X) \tag{3.20}$$

The theory is invariant under gauge transformations $\delta\phi = 0$, $\delta\psi =$ arbitrary, so we can always choose a gauge ("Siegel gauge") in which $\psi = 0$ (equivalently, $b\Phi = 0$). Then a two-point function in this gauge will have the form

$$\langle\!\langle \Phi(X_f, b_f)\Phi(X_i, b_i)\rangle\!\rangle = b_f b_i \langle\!\langle \phi(X_f)\phi(X_i)\rangle\!\rangle \tag{3.21}$$

(Double brackets denote second-quantized expectation values.) The "reduced two-point function" $\langle\!\langle \phi(X_f)\phi(X_i)\rangle\!\rangle$ is the inverse of $\Delta + m^2$, where Δ is the D-dimensional Laplacian, $(\partial/\partial X^\mu)(\partial/\partial X^\mu) = \Delta$:

$$\begin{aligned}\langle\!\langle \phi(X_f)\phi(X_i)\rangle\!\rangle &= \frac{1}{(\Delta + m^2)(X_f, X_i)} \\ &= \int \frac{d^D k}{(2\pi)^{D/2}} \frac{\exp[ik^\mu (X_f^\mu - X_i^\mu)]}{k^\mu k^\mu + m^2}\end{aligned} \tag{3.22}$$

But this is precisely equal to $\langle X_f | X_i\rangle$ of equation (2.64) (see, e.g., Cohen *et al.*, 1986). We conclude that the Polyakov path integral for the point particle gives field-theoretic amplitudes in the Siegel gauge:

$$\langle X_f, b_f | X_i, b_i\rangle = b_f b_i \langle\!\langle \phi(X_f)\phi(X_i)\rangle\!\rangle \tag{3.23}$$

The single Grassmann parameter b here plays the role of the antighost zero mode in the open string.

4. SEWING

By "sewing," we mean the process of constructing the path integral over a single geometrical region from the path integrals over smaller pieces which make up the original region. This process is quite familiar in the nonrelativistic case; the path integral for a particle to go from position $\mathbf{X}_1$ at time t_1 to $\mathbf{X}_3$ at t_3 is the product of the path integrals to go from $\mathbf{X}_1$ at t_1 to $\mathbf{X}_2$ at an intermediate time t_2, times the amplitude to go from $\mathbf{X}_2$ at t_2 to $\mathbf{X}_3$ at t_3, integrated over all possible $\mathbf{X}_2$. That is,

$$\langle \mathbf{X}_3, t_3 | \mathbf{X}_1, t_1\rangle = \int d\mathbf{X}_2 \langle \mathbf{X}_3, t_3 | \mathbf{X}_2, t_2\rangle\langle \mathbf{X}_2, t_2 | \mathbf{X}_1, t_1\rangle \tag{4.1}$$

In the relativistic case, the naive analogue is certainly *not* true:

$$\langle X_3 | X_1\rangle \neq \int d^D X_2 \langle X_3 | X_2\rangle\langle X_2 | X_1\rangle \tag{4.2}$$

where $X = (\mathbf{X}, t)$. The amplitude which *can* be sewn in this manner is the "Schwinger–DeWitt" amplitude, the integrand of (2.64):

$$\langle X_f, \lambda | X_i, 0\rangle = \lambda^{-D/2} \exp\left[-\frac{1}{2\lambda}(X_i - X_f)^2 - \frac{\lambda m^2}{2}\right] \tag{4.3}$$

We have written the left-hand side to (4.3) this way because it can be interpreted as proportional to the amplitude for a free nonrelativistic particle of unit mass to propagate, in time $i\lambda$, from "position" $X_i^\mu = (\mathbf{X}_i, t_i)$ to "position" $X_f^\mu = (X_f, t_f)$ (DeWitt, 1964; Brown, 1977); up to X- and λ-independent factors,

$$\langle X_f, \lambda \,|\, X_i, 0\rangle = \left[\exp\left(-\frac{\lambda m^2}{2}\right)\right] \langle X_f|\exp(-\lambda \hat{H})|X_i\rangle \tag{4.4}$$

where $\hat{H}$ is the first-quantized Hamiltonian operator for this fictitious particle,

$$\hat{H} = \frac{1}{2}\frac{\partial}{\partial X^\mu}\frac{\partial}{\partial X^\mu} = \frac{1}{2}\Delta \tag{4.5}$$

and the $|X\rangle$ are the Schrödinger-picture D-dimensional position eigenstates. This amplitude satisfies a relation analogous to the nonrelativistic relation (4.1),

$$\langle X_3, \lambda_1+\lambda_2 | X_1, 0\rangle = \int d^D X_2 \langle X_3, \lambda_3 | X_2, \lambda_2\rangle\langle X_2, \lambda_2 | X_1, 0\rangle \tag{4.6}$$

as can be seen using (4.4) and the completeness relation

$$\int d^D X\, |X\rangle\langle X| = 1 \tag{4.7}$$

5. THE VACUUM AMPLITUDE

In second-quantized field theory, a quantity which is often of importance is the vacuum amplitude, the trace of the logarithm of the second variational derivative of the second-quantized action (see, e.g., Abers and Lee, 1973; Ramond, 1981; Ryder, 1985). For a scalar field, corresponding to a spinless point particle, this second variational derivative is $(\Delta+m^2)(X_2, X_1)$ and the vacuum amplitude is

$$A_{\text{vac}} = \operatorname{tr}\log(\Delta+m^2) = \int d^D X \log[(\Delta+m^2)(X, X)] \tag{5.1}$$

This is usually represented, in terms of Feynman diagrams, as a closed loop. This picture might suggest that A_{vac} can be obtained simply by sewing together the ends of a single propagator; that is, performing the first-quantized path integral over closed loops going from X^μ back to X^μ, then integrating over X^μ. This is, of course, incorrect, since such a procedure would give $\operatorname{tr}(\Delta+m^2)^{-1} = \int d^D X\, [(\Delta+m^2)^{-1}(X, X)]$ rather than (5.1).

What is remarkable is that, if one performs the path integral over closed loops directly, one *does* obtain (5.1). The analogous property has been shown for the corresponding closed-string diagram, the torus (Polchinski, 1986; Shapiro, 1972); here we show that this is not a peculiarity of string theory, but occurs as well in the particle case.

As in (2.8),

$$\langle \text{loop} \rangle = \int_{\substack{\text{closed} \\ \text{loops in } X}} \frac{Dg_{ab}\, Dx}{V_{GC}} \exp(-S_P) \tag{5.2}$$

The treatment of the integration over worldline metrics is the same as for the propagator, except that the boundary condition (2.21) on diffeomorphism vector fields $\eta^a(\tau)$ is replaced by the weaker condition of periodicity,

$$\eta^a(0) = \eta^a(1) \tag{5.3}$$

Therefore, there are normal modes proportional both to $\cos(2\pi m\lambda)$, $m = 0, 1, 2, \ldots$, and $\sin(2\pi m\lambda)$, $m = 1, 2, \ldots$. The eigenvalues corresponding to both of these are

$$\tilde{\Lambda}_m \equiv (2\pi m/\lambda)^2 \tag{5.4}$$

There is a zero mode; the cosine mode with $m = 0$ violates (2.25). Using the letter M to indicate both the cosine modes and the sine modes, and denoting the zero mode by $M = 0$, the analogue of equation (2.39) for the propagator becomes

$$\begin{aligned} \langle \text{loop} \rangle &= \int_0^\infty d\lambda \left[\frac{\int \prod'_M d\eta^a_M}{V_{GC}} \right] \lambda^{-1/2} \left(\prod_M{}' \tilde{\Lambda}_M^{1/2} \right) \int_{\substack{\text{closed} \\ \text{loops in } X}} DX\, e^{-S_P} \\ &= \int_0^\infty d\lambda \left[\frac{1}{\int d\eta^a_0} \right] \lambda^{-1/2} \left(\prod_{m=1}^\infty \frac{2\pi m}{\lambda} \right)^2 \int_{\substack{\text{closed} \\ \text{loops in } X}} DX\, e^{-S_P} \end{aligned} \tag{5.5}$$

since $V_{GC} = \int \prod_M d\eta^a_M$, *including* $M = 0$ [see discussion following equation (2.39)]. The normalized $M = 0$ eigenfunction is $\lambda^{-3/2}$ [see (2.34)], so the change in $\eta^a(\tau)$ due purely to a change in the mode η^a_0 is

$$d\eta^a(\tau)_{ZM} = \lambda^{-3/2}\, d\eta^a_0 \tag{5.6}$$

or

$$d\eta^a_0 = \lambda^{3/2}\, d\eta^a(\tau)_{ZM} \tag{5.7}$$

[Clearly, $d\eta^a(\tau)_{ZM}$ is independent of τ, since λ and η^a_0 are.] $\eta^a(\tau)$ is defined with no reference to the fiducial metric (Lie dragging is a metric-independent concept.) So, if we use (5.7) in (5.5) to write

$$\langle\text{loop}\rangle = \int_0^\infty d\lambda\, \lambda^{-2}\left(\prod_{m=1}^{\infty} \frac{2\pi m}{\lambda}\right)^2 \int_{\substack{\text{closed}\\ \text{loops in } X}} DX\, e^{-S_P} \tag{5.8}$$

we have left out a factor, $\int d\eta^a(\tau)_{\text{ZM}}$, which is *independent* of λ; but we have been dropping such factors all along. (The reason for our focus on the λ dependence of the integrand will be discussed at the end.)

Modifications similar to those just encountered in the η_a integration are found to occur in the X^μ integration for the loop. In contrast to the propagator calculation, we are here integrating over paths which are constrained only by the requirement of periodicity,

$$X^\mu(0) = X^\mu(1) \tag{5.9}$$

The "classical paths", i.e., paths satisfying (2.45) as well as (5.9), are all functions of the form $X^\mu(\tau) = \text{const}$. Note, however, that these paths cannot simultaneously satisfy the classical equation of motion (2.7) for g_{ab}. Rather than expanding about a classical path, we can simply do the path integral over $X^\mu(\tau)$ directly without making the separation (2.44) into classical and quantum parts. As we did with η^a, we expand the function $X^\mu(\tau)$ in a full-range Fourier series, with normal modes proportional to $\cos(2\pi m\lambda)$, $m = 0, 1, \ldots$, and $\sin(2\pi m\lambda)$, $m = 1, 2, \ldots$. Again using the index M to denote all the modes, with $M = 0$ corresponding to the $m = 0$ cosine mode, and with the prime on sums and products indicating deletion of the zero mode,

$$\begin{aligned} &\int_{\substack{\text{closed}\\ \text{loops in } X}} DX\, e^{-S_P} \\ &= \prod_{\mu=1}^{D} \left(\int dX_0^\mu \prod_M{}' \int dX_M^\mu\right) \exp\left[-\frac{\lambda m^2}{2} - \frac{1}{2}\sum_M{}' \left(\frac{2\pi M}{\lambda}\right)^2 X_M^\mu X_M^\mu\right] \\ &= \left[\exp\left(-\frac{\lambda m^2}{2}\right)\right]\left(\prod_{m=1}^{\infty} \frac{2\pi m}{\lambda}\right)^{-2D} \prod_{\mu=1}^{D} \int dX_0^\mu \end{aligned} \tag{5.10}$$

The factor $\prod_{\mu=1}^{D} \int dX_0^\mu$ is an infinity proportional to the volume of D-dimensional spacetime. It is to be expected, since the closed loops over which we are summing can start and end anywhere. However, the coefficient of the zero mode is λ dependent; in fact, the normalized zero eigenmode is $\lambda^{-1/2}$ [see (2.51)]. So, the change in $X^\mu(\tau)$ due to a pure zero-mode deformation is

$$dX^\mu(\tau)_{\text{ZM}} = \lambda^{-1/2}\, dX_0^\mu \tag{5.11}$$

or

$$dX_0^\mu = \lambda^{1/2}\, dX^\mu(\tau)_{\rm ZM} \tag{5.12}$$

Using (5.10) and (5.12) in (5.8), we find

$$\langle \text{loop} \rangle = \int_0^\infty d\lambda\, \lambda^{-2+D/2} \left(\prod_{m=1}^{\infty} \frac{2\pi m}{\lambda} \right)^{2-2D} e^{-\lambda m^2/2} \prod_{\mu=1}^{D} \int dX^\mu(\tau)_{\rm ZM} \tag{5.13}$$

The factor $\prod_{\mu=1}^{D} \int dX^\mu(\tau)_{\rm ZM}$ is λ independent, so it can be taken outside the λ integral sign and ignored, along with the many factors of $(2\pi)^\infty$, etc., we have dropped along the way. Applying the zeta-function regularization (2.63) to the infinite product in (5.13),

$$\langle \text{loop} \rangle = \int_0^\infty d\lambda\, \lambda^{-1-D/2}\, e^{-\lambda m^2/2} \tag{5.14}$$

We should remind the reader that, although the λ-factor gymnastics we have performed in the presence of zero modes were compulsory exercises as a consequence of the normalizations we chose for the eigenmodes, these particular normalizations were chosen, not out of love of exercise, but, rather, because we defined our path integral measures in a reparametrization-invariant manner. And we *had* to define them in a reparametrization-invariant manner so as to be able to extract the infinite factor V_{GC}. Any other reparametrization-invariant definition of the path-integral measures would have led to the same λ dependence of the final integrand.

How does (5.14) compare with "sewing" the propagator? From (2.64) or (4.3),

$$\begin{aligned} \text{tr}\langle X_i | X_f \rangle &= \int d^D X\, \langle X | X \rangle = \int d^D X \int_0^\infty d\lambda\, \langle X, \lambda | X, 0 \rangle \\ &= \int_0^\infty d\lambda\, \lambda^{-D/2}\, e^{-\lambda m^2/2} \end{aligned} \tag{5.15}$$

Or, using (2.64) and (4.3)–(4.5),

$$\begin{aligned} \text{tr}\langle X_i | X_f \rangle &= \int d^D X \int_0^\infty d\lambda\, \langle X | e^{-(\lambda/2)(\Delta + m^2)} | X \rangle \\ &= \int d^D X \left\langle X \left| \frac{1}{\Delta + m^2} \right| X \right\rangle \\ &= \int d^D X\, [(\Delta + m^2)^{-1}(X, X)] \end{aligned} \tag{5.16}$$

[See, e.g., Dirac (1958) regarding matrix elements of operators.] From the representation of equation (5.15) it is clear that ⟨loop⟩ in (5.14) is a different quantity, due to the extra factor of λ^{-1} in the integrand. What is the operator representation of $\langle\text{loop}\rangle$, corresponding to the representation (5.16) of $\text{tr}\langle X_i | X_f\rangle$? We have

$$\log[(\Delta+m^2)(X,X)] = \int_0^\infty d\lambda\, \lambda^{-1}\, e^{-\lambda(\Delta+m^2)(X,X)}$$
$$= \int_0^\infty d\lambda\, \lambda^{-1}\langle X| e^{-\lambda(\Delta+m^2)}|X\rangle$$
$$= \int_0^\infty d\lambda\, \lambda^{-1}\langle X, \lambda\,|\, X, 0\rangle \tag{5.17}$$

using (4.4) and (4.5). If we next use (4.3) in the above, we obtain

$$\log[(\Delta+m^2)(X,X)] = \int_0^\infty d\lambda\, \lambda^{-1-D/2}\, e^{-\lambda m^2/2} \tag{5.18}$$

Comparing (5.18) with (5.1) and (5.14), we see that the Polyakov path integral over loops has indeed given us the field-theoretic vacuum amplitude:

$$\langle\text{loop}\rangle = A_{\text{vac}} \tag{5.19}$$

[In fact, the same infinite constant, $\prod_{\mu=1}^{D} \int dX_{\text{ZM}}^{\mu}$ = volume of spacetime, which we discard in computing $\langle\text{loop}\rangle$, must also be removed when evaluating A_{vac}, since the integrand on the right-hand side of (5.1) is explicitly independent of X^μ.

The reader may well object to all this to-do regarding the "equality" (5.19) of two quantities each of which is computed with total disregard of so many factors. Indeed, (5.19) is *divergent*, due to the behavior of the integrand at $\lambda = 0$ (ultraviolet divergence). To the extent that (5.19) makes sense in and of itself, it is in the equivalent dependence of the left- and right-hand sides on the mass m. (This in turn is controlled by the λ dependence of the respective integrands; hence our focus on that aspect of the computation.) In string theory, the relation corresponding to (5.19), and also the actual value of the vacuum amplitude, are of central importance; it is interesting to see that this relation, as well as the other relations obtained in this paper, hold as well for the point particle.

APPENDIX

In equation (2.10), we have $\int D\hat{g}\, DX\, D\eta$ rather than, say, $\int D\hat{g}\, D\hat{X}\, D\eta$, [the set of $\hat{X}^\mu(\tau)$ being the set of $X^\mu(\tau)$ not related by diffeomorphism.] That is, in writing (2.10), we are claiming that the space of ordered pairs $(g_{ab}(\tau), X^\mu(\tau))$ not related by diffeomorphisms is isomorphic to the space of ordered pairs of the form $(\hat{g}_{ab}(\tau), X^{\mu}(\tau))$, where $\hat{g}_{ab}(\tau)$ are metrics not related by diffeomorphisms but where the $X^{\mu}(\tau)$ are arbitrary.

The situation here is analogous to a system consisting of two mass points connected to each other by a spring, but subject to no external forces. Let us use the term "gauge group" to refer to a group G of transformations which leaves the energy of the system invariant. For G, choose the group of translations. The elements of G are vectors $\mathbf{a}$. The configuration space Q of the system is the space consisting of the ordered pairs of the position vectors of the two mass points $(\mathbf{q}_1, \mathbf{q}_2)$. The entire configuration space includes configurations related by G, e.g., $(\mathbf{q}_1, \mathbf{q}_2)$ and $(\mathbf{q}_1', \mathbf{q}_2') = (\mathbf{q}_1 + \mathbf{a}, \mathbf{q}_2 + \mathbf{a})$. The space of configurations *not* related by gauge transformations, which we will call $\hat{Q}$, is isomorphic to the space of ordered pairs with one element, say the first, *fixed*, and the second arbitrary:

$$\hat{Q} = \{(\mathbf{q}_{\text{fixed}}, \mathbf{q})\} \tag{A1}$$

This is true because two conditions are met: every element of Q can be obtained from some element of $\hat{Q}$; and no element of Q can be obtained from more than one element of $\hat{Q}$.

To prove that the first condition is met, let $(\mathbf{q}_1, \mathbf{q}_2)$ be an arbitrary element of Q. Since the second entry of an element of $\hat{Q}$ is arbitrary, one such element is $(\mathbf{q}_{\text{fixed}}, \mathbf{q}_3)$, where $\mathbf{q}_3 = \mathbf{q}_2 + \mathbf{q}_{\text{fixed}} - \mathbf{q}_1$. Then $(\mathbf{q}_1, \mathbf{q}_2)$ can be obtained from $(\mathbf{q}_{\text{fixed}}, \mathbf{q}_3)$ by the gauge transformation (translation) $\mathbf{a} = \mathbf{q}_1 - \mathbf{q}_{\text{fixed}}$,

$$(\mathbf{q}_1, \mathbf{q}_2) = (\mathbf{q}_{\text{fixed}} + \mathbf{a}, \mathbf{q}_3 + \mathbf{a}) \tag{A2}$$

If we had taken $\hat{Q}$ to be the space of ordered pairs with *both* entries fixed, i.e., the single pair of points $(\mathbf{q}_{\text{fixed}}, \mathbf{q}_{\text{fixed}}')$, we could not obtain all elements of Q from $\hat{Q}$ using gauge transformations, since not all elements $(\mathbf{q}_1, \mathbf{q}_2)$ of Q are of the form $(\mathbf{q}_{\text{fixed}} + \mathbf{a}, \mathbf{q}_{\text{fixed}}' + \mathbf{a})$. (Translations move both mass points by the same amount and do not affect their relative position.) As for the second condition, suppose that one element of Q could be obtained from two different elements of $\hat{Q}$, $(\mathbf{q}_{\text{fixed}}, q_3)$ and $(\mathbf{q}_{\text{fixed}}, q_4)$, where $\mathbf{q}_3 \neq \mathbf{q}_4$. That is,

$$(\mathbf{q}_1, \mathbf{q}_2) = (\mathbf{q}_{\text{fixed}} + \mathbf{a}, \mathbf{q}_3 + \mathbf{a}) \tag{A3}$$

$$(\mathbf{q}_1, \mathbf{q}_2) = (\mathbf{q}_{\text{fixed}} + \mathbf{a}', \mathbf{q}_4 + \mathbf{a}') \tag{A4}$$

Subtracting (A4) from (A3), we find

$$\mathbf{q}_{\text{fixed}} + \mathbf{a} = \mathbf{q}_{\text{fixed}} + \mathbf{a}' \tag{A5}$$

$$\mathbf{q}_3 + \mathbf{a} = \mathbf{q}_4 + \mathbf{a}' \tag{A6}$$

from which we conclude $\mathbf{q}_3 = \mathbf{q}_4$, in contradiction to our assumption. So the second condition is also satisfied.

[We could *not* reach this conclusion here if the gauge group of translations did not act *effectively*, i.e., if there were some pairs of points $(\mathbf{q}_1, \mathbf{q}_2)$ left unchanged by nonzero transformations $\mathbf{a} \neq 0$. This would be the case if, instead of translations, we had considered rotations (they leave the origin unchanged); it is also the case when considering diffeomorphisms of the loop, since uniform translations $\tau \to \tau + \text{const}$ leave both $g_{ab}(\tau)$ and $X^\mu(\tau)$ unchanged. For a discussion of this more complicated situation, see Moore and Nelson (1986).]

The main difference between the two mass-point system $(\mathbf{q}_1, \mathbf{q}_2)$ with gauge group translations and the $(g_{ab}(\tau), X^\mu(\tau))$ system with gauge group diffeomorphisms is that in the latter the space of "first entries" not related to each other by gauge transformations is not just a single point $\mathbf{q}_{\text{fixed}}$, but rather the one-parameter set of fiducial metrics:

$$\text{“}Q\text{”} = (g_{ab}(\tau), X^\mu(\tau)) \tag{A7}$$

$$\text{“}\hat{Q}\text{”} = (\lambda, X^\mu(\tau)) \tag{A8}$$

In the $(\mathbf{q}_1, \mathbf{q}_2)$ system we could equally well have fixed the second entry rather than the first (or some combination such as the center of mass). In the $(g_{ab}(\tau), X^\mu(\tau))$ system we could in principle find a set of "Teichmuller X's" $\hat{X}^\mu(\tau)$; but $g_{ab}(\tau)$ does not enter the action quadratically (it does not even enter as a polynomial!), so we would be at a loss as to how to perform the resulting path integral over all metrics.

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